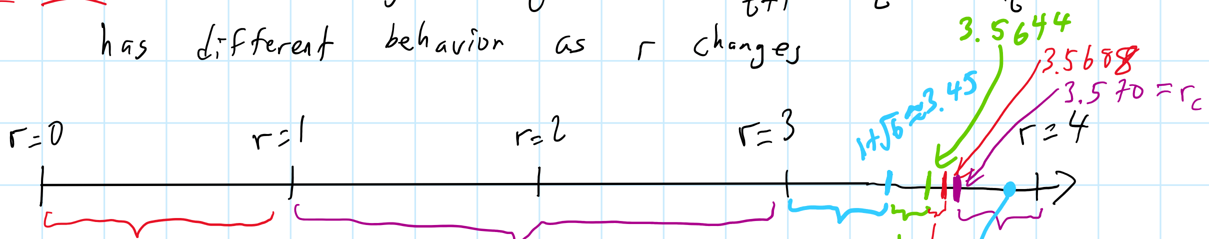
2.7a bifurcation theory

Wednesday, February 3, 2021 2:01 PM

Last time: discrete logistic equation $x_{t+1} = r x_t (1 - x_t)$ has different behavior as r changes



- $r=0$ to $r=1$: 0 = stable fixed pt ($\frac{r-1}{r} < 0$ unstable)
- $r=1$ to $r=3$: $\frac{r-1}{r}$ stable fixed pt; 0 unstable fixed pt
- $r=3$ to $1+\sqrt{6}$: $0, \frac{r-1}{r}$ unstable; stable 2-cycle
- stable 4-cycle
- stable 8-cycle
- chaotic behavior for most values "uncountably infinite number of trajectories sometimes close close together and sometimes far apart"
- $r \approx 3.828$ is an island of stability with a 3-cycle

A **bifurcation** happens when the behavior of a system changes as you go past a particular value of a parameter.

More formally, consider $x_{t+1} = f(x_t, r)$

Then for any value of r, there is a "set" of equilibria denoted $\bar{x}(r)$. e.g. $\bar{x}(r) = \{0, \frac{r-1}{r}\}$

(Sometimes, we also include m-cycles in the set $\bar{x}(r)$)

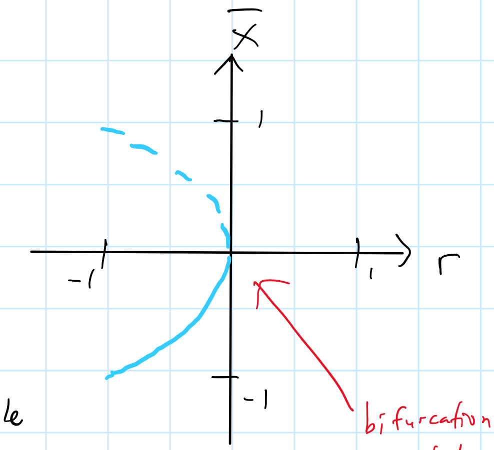
The values of r where $\bar{x}(r)$ changes are **bifurcation values** $\bar{r}$.

and the points $(\bar{r}, \bar{x}(\bar{r}))$ are the **bifurcation points**.

$\bar{x}(r)$ changes when equilibrium or m-cycle changes behavior, i.e. when the derivative of f or f^m has absolute value 1.

Define: A **bifurcation diagram** plots $\bar{x}(r)$ as a function of r, using a solid line for stable equilibria (or n-cycles) and a dashed line for unstable equilibria (or n-cycles).

Ex. $x_{t+1} = r + x_t + x_t^2$, $\quad -1 < r < 1$



Then $\bar{x} = r + \bar{x} + \bar{x}^2$

$\Rightarrow \quad -r = \bar{x}^2$

$\Rightarrow \quad \bar{x} = \pm\sqrt{-r}$ if $r < 0$ ← complex if $r > 0$

Note $f(x) = r + x + x^2$

$f'(x) = 1 + 2x$

$f'(\sqrt{-r}) = 1 + 2\sqrt{-r} > 1$ for $r < 0$, so unstable

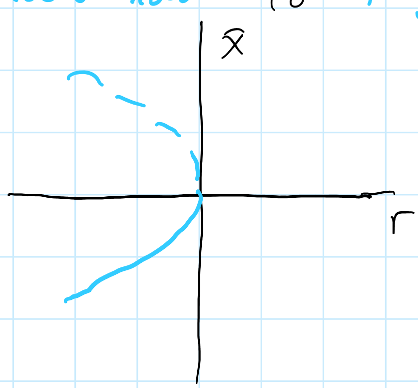
$f'(-\sqrt{-r}) = 1 - 2\sqrt{-r}$

$|f'(-\sqrt{-r})| < 1$ if $r > -1$, so stable

$|f'(-\sqrt{-r})| > 1$ if $r < -1$, so unstable

Classification of bifurcations where $f'(\bar{x}(\bar{r})) = \pm 1$.

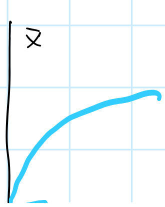
I. Saddle node (or tangent) "blue-sky bifurcation" $\quad f'(\bar{x}(\bar{r})) = 1$

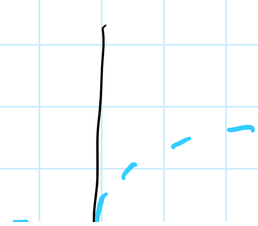


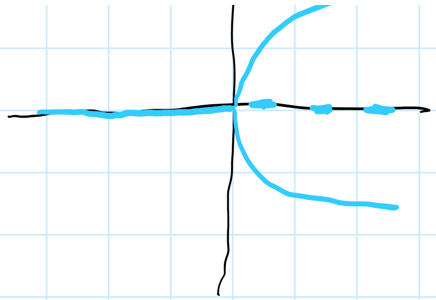
A pair of equilibria, one stable, one unstable, (dis)appear together.

Ex. $x_{t+1} = r + x_t + x_t^2$, at $r = 0$ as shown above.

II. Pitchfork $\quad f'(\bar{x}(\bar{r})) = 1$

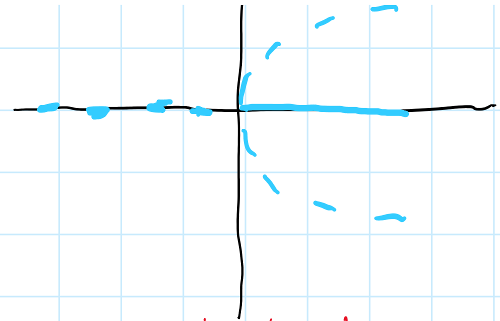






supercritical

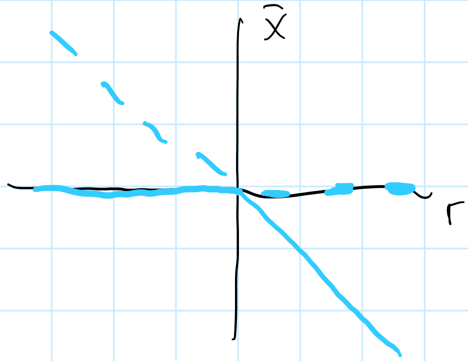
A stable fixed pt becomes two stable fixed pts separated by an unstable fixed pt.



subcritical

An unstable fixed pt becomes 2 unstable fixed pts separated by a stable fixed pt.

III. Transcritical bifurcation



Two fixed pts of opposite stability exchange as the bifurcation pt is passed.

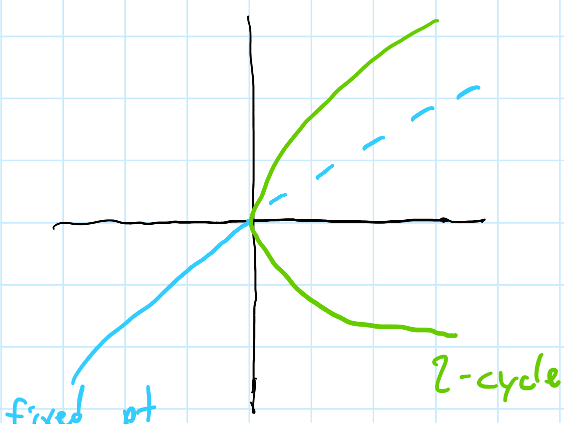
Ex. $x_{t+1} = r x_t (1 - x_t)$ at $r = 1$ $\qquad f'(\bar{x}(\bar{r})) = 1$

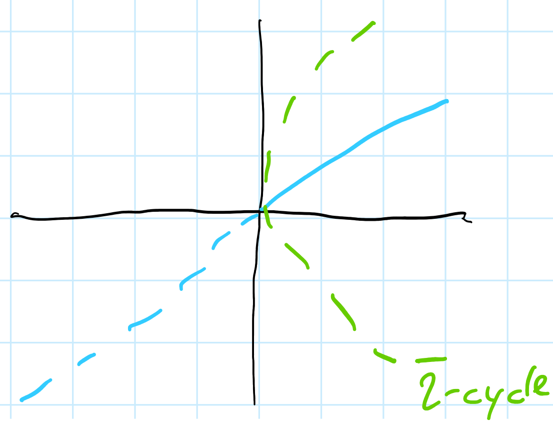
Above when $0 < r < 1$, 0 is stable and $\bar{x} = \frac{r-1}{r}$ is negative but also unstable

For $1 < r < 3$, 0 is unstable and $\bar{x} = \frac{r-1}{r}$ is stable

IV. Period doubling / flip bifurcation

$f'(\bar{x}(\bar{r})) = -1$





supercritical

A stable fixed pt becomes unstable as a stable 2-cycle appears

subcritical

An unstable fixed pt becomes a stable fixed pt as an unstable 2-cycle appears

Aside: The first bifurcations also appear in differential equations, but the period doubling bifurcation is unique to difference equations.

Ex. $x_{t+1} = r x_t (1 - x_t)$ at $r = 3$.

Ex. $x_{t+1} = r + x_t + x_t^2$ when $r = -1$

equilibria: $-i\sqrt{r}$, $i\sqrt{r}$

2-cycle: $-i\sqrt{r+1} - 1$, $i\sqrt{r+1} - 1$

$f(x) = r + x + x^2$

$f'(x) = 1 + 2x$, so $|f'(-i\sqrt{r})| > 1$ if $r < -1$

When $r < -1$, the 2-cycle is real.

Also, $|f'(-i\sqrt{r+1} - 1)\, f'(i\sqrt{r+1} - 1)|$

$$= \left| [1 + 2(-i\sqrt{r+1} - 1)][1 + 2(i\sqrt{r+1} - 1)] \right|$$

$$= \left| (-1 - 2i\sqrt{r+1})(-1 + 2i\sqrt{r+1}) \right|$$

$$= |1 + 4(r+1)| = |5 + 4r| \begin{cases} < 1 & \text{when } -\frac{3}{2} < r < -1 \\ > 1 & r < -\frac{3}{2} \end{cases}$$

Then a stable fixed pt turns unstable and a stable 2-cycle appears at $r = -1$.